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LONG-WAVE–SHORT-WAVE INTERACTION IN BUBBLY LIQUIDS†

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The interaction of long and short waves in a rarefied monodisperse mixture of a weakly compressible liquid containing bubbles of gas is considered. It is shown that the equations describing the dynamics of the perturbations in the bubbly liquid admit of the existence of short-wave–long-wave Benney–Zakharov resonance. A special modification of the multiple-scale method is employed to derive the interaction equations. In the non-resonant case, the interaction equations reduce to the non-linear Schrödinger equation in the form of the short-wave envelope while, in the resonance case, they reduce to the well-known system of Zakharov equations. The characteristics of long-wave–short-wave interaction in a bubbly liquid lie in the fact that, at certain values of the frequency of the short wave, the interaction coefficients vanish ("interaction degeneracy"). A class of new interaction models is constructed in the case of "degeneracy". Degenerate resonance interaction in a bubbly liquid is investigated numerically using these models. © 2000 Elsevier Science Ltd. All rights reserved.

It is well known that the independent propagation of long and short waves on the water surface and in a collision-free plasma at fairly long times can be described, respectively, by the Korteveg–de Vries [1, 2] equation

$$\frac{\partial L}{\partial t}+\sigma\frac{\partial L^2}{\partial x}+\chi\frac{\partial^3 L}{\partial x^3}=0$$

and the non-linear Schrödinger equation [3, 4]

$$i\frac{\partial S}{\partial t}+\beta\frac{\partial^2 S}{\partial x^2}+\gamma|S|^2 S=0$$

(L is the profile of the long wave and S is the envelope of the short waves).

When the amplitude of the short-wave signal varies with time and in space, interaction can occur between the long wave and the envelope of the short waves. The mechanism of "long-wave–short-wave" interaction was investigated for the first time in [5] when studying waves on the surface of water. A general theory of the interaction between long and short waves was proposed in [6], where a new form of resonance between three waves with wave numbers k_1, k_2, k_3 and frequencies ω_1, ω_2, ω_3 was considered. Actually, by choosing

$$k_1=k_s+\varepsilon k',\quad k_2=k_s-\varepsilon k',\quad k_3\equiv k_l=2\varepsilon k',\quad k_s,k'=O(1),\quad \varepsilon\ll 1$$

the resonance condition $k_1 = k_2 + k_3$, $\omega_1 = \omega_2 + \omega_3$ can be reduced to the form $d\omega_s/dk_s = \omega_l/k_l$. If k_s and ω_s are the wave number and frequency of the short wave and k_l and ω_l are the wave number and frequency of the long wave, the condition which has been obtained assumes that the group velocity of the short wave $c_g(k_s) = d\omega_s/dk_s$ and the phase velocity of the long wave $c_p(k_l) = \omega_l/k_l$ are equal (long-wave–short-wave resonance).

The equations of long-wave–short-wave resonant interaction

$$\frac{\partial L}{\partial t}+\alpha\frac{\partial |S|^2}{\partial x}=0,\quad i\frac{\partial S}{\partial t}+\beta\frac{\partial^2 S}{\partial x^2}=\delta LS$$

were proposed for the first time by Zakharov to describe the interaction of Langmuir oscillations with ionic sound in a plasma [4]. These universal equations were obtained for waves on the water surface [7] and for the model of a molecular chain in the form of an α-helix [8].

Due to the presence of gas bubbles, the dispersion curve for a bubbly liquid separates into two branches, a low-frequency branch and a high-frequency branch [9]. The mutual propagation of long-wave and short-wave perturbations is therefore possible in such a medium. However, investigations, using the theory of non-linear waves, of liquids containing gas bubbles have been confined to an independent treatment of the long and short waves. In particular, a Korteveg–de Vries equation was obtained in [10, 11] to describe the evolution of long-wave

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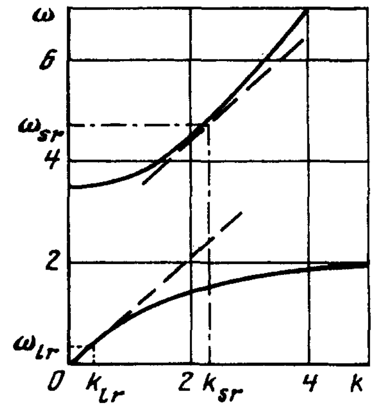

Fig. 1.

perturbations in liquids with bubbles of adiabatic gas. It has been shown [12] that the evolution of quasi-monochromatic wave packets of short waves in polydisperse bubbly liquids can be described by the non-linear Schrödinger equation. Equations have been constructed for the modulations of the short waves in the case of a bubbly mixture with an incompressible carrier phase [13] and, also, for the case when interphase heat exchange is taken into account [14]. The papers [15, 16] should also be noted in which models of the generation of the subharmonics of short waves were proposed and the effects of parametric amplification associated with them and the generation of sound in liquids containing gas bubble were analysed.

In this paper, the interaction of one-dimensional long-wave and short-wave perturbations in a monodisperse bubbly liquid is investigated for the first time.

1. BASIC EQUATIONS

The one-dimensional flow of an ideal, slightly compressible liquid containing a small amount of spherical gas bubbles, under conditions where heat dissipation and capillary effects can be neglected, is described by the equations [9, 17, 18]

$$\frac{d\rho}{dt}+\rho\frac{\partial v}{\partial x}=0,\quad \rho\frac{dv}{dt}+\frac{\partial p}{\partial x}=0,\quad \frac{dn}{dt}+n\frac{\partial v}{\partial x}=0$$

$$\rho_l\left[a\frac{d^2a}{dt^2}+\frac{3}{2}\left(\frac{da}{dt}\right)^2\right]=p_g-p,\quad \rho_l-\rho_{l0}=\frac{p-p_0}{C_l^2} \tag{1.1}$$

$$\rho=\rho_l(1-\alpha_g),\quad \alpha_g=\frac{4}{3}\pi a^3 n,\quad \frac{p}{p_0}=\left(\frac{a_0}{a}\right)^{3\varkappa}$$

Here $d/dt=\partial/\partial t+v\partial/\partial x$ is a substantive derivative with respect to time, p, ρ, v are the pressure, density and velocity of the mixture, ρ_l is the true density of the liquid, C_l is the speed of sound in the pure liquid, p_g, α_g and a are the pressure, the volume content and radius of the bubbles, n is the number of bubbles per unit volume of the mixture, $\varkappa$ is the polytropy exponent, and a zero subscript denotes the unperturbed state of the mixture.

Changing to the dimensionless quantities

$$\bar a=\frac{a}{a_0}-1,\quad \bar p=\frac{p}{p_0}-1,\quad \bar\rho=\frac{\rho-\rho_0}{\rho_*},\quad \bar v=\frac{v}{v_*},\quad \bar t=\frac{t}{t_*},\quad \bar x=\frac{x}{l_*}$$

$$\rho_*=\rho_0\alpha_{g0},\quad v_*=\sqrt{\frac{\alpha_{g0}p_0}{\rho_0}},\quad t_*=a_0\sqrt{\frac{\rho_0}{p_0}},\quad l_*=\frac{a_0}{\sqrt{\alpha_{g0}}}$$

(the bar is subsequently omitted) and neglecting quantities of the order of the volume gas content compared with unit, system (1.1) can be reduced to the form [12]

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0, \quad (1+a)\frac{d^2 a}{dt^2} + \frac{3}{2}\left(\frac{da}{dt}\right)^2 - (1+a)^{-3\varkappa} + p + 1 = 0 \tag{1.2}$$

$$\rho - 1 - b^2 p + (1+a)^3 = 0, \quad b = \sqrt{p_0(\rho_{l0}\alpha_{g0}C_l^2)^{-1}}$$

A linear analysis of system (1.2) shows that its dispersion relation has two branches

$$\omega_\pm^2(k) = \frac{1}{2}\{3\varkappa + (k^2+3)b^{-2} \pm \sqrt{[3\varkappa - (k^2+3)b^{-2}]^2 + 36\varkappa b^{-2}}\}. \tag{1.3}$$

The dispersion curve (1.3) is shown in Fig. 1 for a mixture of water with air bubbles under normal conditions ($p_0 = 0.1$ MPa and $\rho_{l0} = 10^3$ kg/m^3) and a volume gas content $\alpha_{g0} = 1.1 \times 10^{-4}$.

The long-wave and short-wave asymptotic forms of dispersion relation (1.3) are as follows:

$$\begin{aligned} &\omega_l = c_e k_l - \chi k_l^3 + O(k_l^5) \quad \text{for} \quad k_l \to 0; \quad \chi = c_e^5/(6\varkappa^2) \\ &\omega_s = c_f k_s + O(k_s^{-1}) \quad \text{for} \quad k_s \to \infty \end{aligned} \tag{1.4}$$

where

$$c_e = \omega_l / k_l |_{k_l \to 0} = \sqrt{(b^2 + \varkappa^{-1})^{-1}}, \quad c_f = d\omega_s / dk_s |_{k_s \to \infty} = b^{-1} \tag{1.5}$$

are the equilibrium and frozen speed of sound in the mixture.

The dispersion relation (1.3) admits of the existence of long-wave–short-wave Benney–Zakharov resonance. Actually, since the group velocity of the short wave

$$c_g = d\omega_s / dk_s = k_s \omega_s^{-1}(\omega_s^2 - 3\varkappa)(2b^2\omega_s^2 - 3\varkappa c_e^{-2} - k_s^2)^{-1} \tag{1.6}$$

where k_s tends to zero is an infinitesimal quantity, and it follows from (1.5) that $c_f > c_e$, a $k_s = k_{sr}$ is found for any sufficiently small $k_l = k_{lr}$ such that the long-wave–short-wave Benney–Zakharov resonance condition (Fig. 1)

$$c_g(k_{sr}) = c_p(k_{lr}) \tag{1.7}$$

is satisfied.

In the case of infinitely long waves ($k_l \to 0$), this condition reduces to the equality

$$c_g(k_{sr}) = c_e \tag{1.8}$$

2. THE METHOD OF MULTIPLE SCALES

The method of multiple scales [19] is used to derive the equations for the interaction of long and short waves. This method assumes that the solution $\mathbf{z} = (a, p, \rho)$ of system (1.1) is expanded in powers of a certain small parameter ε (which characterizes the amplitude of a perturbation) into long-wave and short-wave components

$$\mathbf{z} = \varepsilon^l \sum_{m \geq 1} \varepsilon^{m-1} \mathbf{z}_m^{(0)} + \varepsilon^s \sum_{m,n \geq 1} \varepsilon^{(m-1)n+(n-1)s}[\mathbf{z}_m^{(n)} e^{in\Theta} + \text{c.c.}] \tag{2.1}$$

and that fast (t_0, x_0) and slow $(t_n, x_n) = \varepsilon^n(t_0, x_0)$ $(n = 1, 2, \ldots)$ variables are introduced with the substitutions

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t_0} + \sum_{n \geq 1} \varepsilon^n \frac{\partial}{\partial t_n}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x_0} + \sum_{n \geq 1} \varepsilon^n \frac{\partial}{\partial x_n} \tag{2.2}$$

Here, $\Theta = k_s x_0 - \omega_s t_0$ is the phase of the short wave, $\mathbf{z}_m^{(0)}, \mathbf{z}_m^{(n)}$ $(m, n = 1, 2, \ldots)$ are the long-wave and short-wave components of the solution, which depend solely on the slow variables, and l and s are certain numbers which, together with ε characterize the degree of smallness of the amplitudes of the perturbations.

3. NON-RESONANT AND RESONANT INTERACTION

In order to construct a model of the non-resonant interaction, we put $(l, s) = (2, 1)$, substitute (2.1) and (2.2) into (1.2) and split the resulting expression into harmonics. We restrict ourselves to the order of magnitude ε^4 for the zeroth harmonic and ε^3 for the first harmonic and introduce the notation $L = p_1^{(0)}$ for the pressure profile in the long wave and $S = p_1^{(1)}$ for the envelope of the pressure of the short wave. Then, changing to a system of coordinates which moves in a time t_1 with the group velocity of the short wave $c_g(k_s)$, we obtain the equations for the non-resonant interaction

$$\Phi(\xi) = \Phi_0 + \frac{\alpha}{c_g^2 - c_e^2}|S|^2, \quad i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} + \gamma |S|^2 S = \delta \Phi S \tag{3.1}$$

Here

$$L = L_0(\eta_1) + L_0(\eta_2) + \Phi(\xi), \quad \tau = t_2, \quad \xi = x_1 - c_g t_1, \quad \eta_{1,2} = x_1 \pm c_e t_1 \tag{3.2}$$

$$\alpha = -\frac{c_e^2 c_g^2}{\varkappa(\omega_s^2 - 3\varkappa)^2}[\omega_s^2 - 9\varkappa(\varkappa+1)]$$

$$\beta = \frac{c_g}{2k_s}\left[1 + \frac{c_g}{\omega_s^2 - 3\varkappa}\{4k_s\omega_s + c_g(k_s^2 + 3(1+\varkappa b^2) - 6b^2\omega_s^2)\}\right]$$

$$\delta = -\frac{c_g\omega_s^2}{2\varkappa k_s(\omega_s^2 - 3\varkappa)^2}[\omega_s^2 - 9\varkappa(\varkappa+1)] \tag{3.3}$$

$$\gamma = -\frac{[\omega_s^2 - 3\varkappa(3\varkappa+1)]\delta}{(\omega_s^2 - 3\varkappa)^2} - \frac{c_g(b^2\omega_s^2 - k_s^2)}{4k_s\omega_s^2(\omega_s^2 - 3\varkappa)^3}[-\omega_s^4 + 27\varkappa^2(\varkappa+1)\omega_s^2 + 27\varkappa^2(\varkappa+1)^2]$$

(L_0 is the initial distribution of the long wave), c_g is determined from (1.6) and c_e is determined from (1.5).

The resulting model describes the formation of an inertialess long wave with a wave packet of short waves and their subsequent interaction.

The system of equations (3.1) reduces to a non-linear Schrödinger equation in the function $\tilde{S} = S \exp\{-i\delta\Phi_0\tau\}$

$$i\frac{\partial \tilde{S}}{\partial \tau} + \beta\frac{\partial^2 \tilde{S}}{\partial \xi^2} + \gamma' |\tilde{S}|^2 \tilde{S} = 0, \quad \gamma' = \gamma - \frac{\alpha\delta}{c_g^2 - c_e^2} \tag{3.4}$$

In the case of long-wave–short-wave resonance $c_g = c_e$, the first equation of (3.1) no longer holds since the coefficient $\alpha(c_g^2 - c_e^2)^{-1}$ becomes infinite. This means that resonance interaction requires the choice of other values of (l, s). Such a choice can be $(l, s) = (2.3/2)$.

We shall assume that all the unknowns depend solely on the variables $\xi = x_1 - c_g t_1$, $\zeta = x_2 - c_g t_2$, $\tau = t_2$. We substitute expressions (2.1) and (2.2) into system (1.2) with the above mentioned choice of (l, s) and split the resulting expression into harmonics, taking account of terms up to the order of magnitude of ε^5 in the case of the zeroth harmonic and $\varepsilon^{7/2}$ in the case of the first harmonic. Then, satisfying the resonance condition (1.8), we obtain the system of Zakharov equations [4]

$$\frac{\partial L}{\partial \tau} + \frac{\alpha}{2c_e}\frac{\partial |S|^2}{\partial \xi} = 0, \quad i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} = \delta L S \tag{3.5}$$

Here, $L = p_1^{(0)}$, $S = p_1^{(1)}$, and the coefficients α, β and δ are determined from (3.3).

4. DEGENERATION OF THE INTERACTION

Analysis of the coefficients (3.3) shows that the interaction coefficients α and δ simultaneously vanish subject to the condition

$$\omega_s^2 = 9\varkappa(\varkappa+1) \tag{4.1}$$

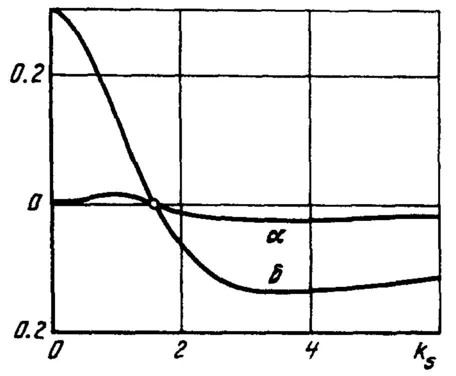

Fig. 2.

In this case, the interaction of the short and long waves "degenerates" since Eqs (3.1) and (3.2) become uncoupled and take the form

$$L = L_0(\eta_1) + L_0(\eta_2), \quad i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} + \gamma |S|^2 S = 0 \tag{4.2}$$

The dependence of the coefficients α and δ on the dimensionless wave number ks is shown in Fig. 2 for a mixture with air bubbles under normal conditions and a volume gas concentration $\alpha_{g0} = 2.2 \times 10^{-4}$. The open circle is the point of degeneracy.

In the non-resonant case, degeneracy leads to a state of affairs where short-wave perturbations will initiate a long wave with a significantly smaller amplitude. This follows from the fact that the associated equations for the interaction can be obtained if one takes the degrees of smallness $(l, s) = (3, 1)$, instead of $(2, 1)$, when substituting expansion (2.1) and (2.2) into Eqs (1.2). On changing the coordinates $x_1 \to \xi = x_1 - c_g t_1$ and using the degeneracy condition (4.1), these equations take the form

$$\Phi(\xi) = i\frac{\lambda}{c_g^2 - c_e^2}\left(S\frac{\partial S^*}{\partial \xi} - S^*\frac{\partial S}{\partial \xi}\right), \quad i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} + \gamma |S|^2 S = 0 \tag{4.3}$$

$$L = L_0(\eta_1) + L_0(\eta_2) + \Phi(\xi), \quad S = S(\tau, \xi), \quad \tau = t_2, \quad \eta_{1,2} = x_1 \pm c_e t_1$$

$$\lambda = (\varkappa + 1)^{1/2} c_e^2 c_g^3 / (3\varkappa^{5/2}(3\varkappa + 2)^2)$$

(the coefficient λ is always non-zero).

In the case of degeneracy of the resonance interaction, that is, when the resonance condition (1.8) and the degeneracy condition (4.1) are satisfied, the system of interaction equations becomes a single parameter equation and is solely determined by the parameter ϰ. First, this follows from degeneracy condition (4.1), according to which the frequency ω_{sr} depends only on ϰ. Second, when the degeneracy condition (4.1) is substituted into dispersion relation (1.3), a relation is obtained which does not contain ω_{sr} and which relates the quantities k_{sr}, b and ϰ. Together with resonance condition (1.8), in which condition (4.1) is also taken into account, this relation constitutes a system of two algebraic equations in k_{sr} and b with the parameter ϰ. Hence, the single free parameter ϰ remains, since the other three (ω_{sr}, k_{sr}, b) are completely expressed in terms of it

$$\omega_{sr} = 3\sqrt{\varkappa(\varkappa + 1)}, \quad k_{sr} = \frac{3(\varkappa + 1)}{\sqrt{\varkappa}(3\varkappa + 2)}, \quad b = \frac{\sqrt{3\varkappa(\varkappa + 1)} + 1}{\varkappa(3\varkappa + 2)} \tag{4.4}$$

If one considers a mixture of water with air bubbles of radius $a_0 = 1$ at a pressure $p_0 = 0, 1$ MPa as the bubbly liquid, then degeneracy of the resonant interaction will occur when a perturbation with a frequency of 985 kHz is initiated, subject to the condition that the volume gas content $\alpha_{g0} \approx 3 \times 10^{-4}$.

We will now consider how resonant interaction equations are modified in the case of degeneracy for different sets of the numbers l and s.

The case $(l, s) = (2, 3/2)$. When condition (4.4) is satisfied, the Zakharov equations (3.5) reduce to a system of linear uncoupled equations

$$\frac{\partial L}{\partial \tau} = 0, \quad i\frac{\partial S}{\partial \tau} + \beta \frac{\partial^2 S}{\partial \xi^2} = 0 \tag{4.5}$$

from which the invariance of the initial profile of the long wave with the passage of time follows. The second equation of (4.5) is the dispersion equation with a dispersion relation of the form $\Omega^2 = \beta^2 K^4$. Due to dispersion of the wave, the packet of the envelope becomes blurred with the passage of time.

System (4.5) admits of a spatially homogeneous solution $L = L_0$, $S = S_0$. Suppose that we specify certain perturbations to these solutions at the initial instant of time. It then follows from Eqs (4.5) that the perturbations of L will propagate without any change in their shape and amplitude but, in the case of S, the perturbations will become blurred over the whole space, decreasing in amplitude for long times. The amplitude of the wave packet of the short waves will decrease as $1/\sqrt{\tau}$, that is, in a time $t_3 = \varepsilon\tau$ the amplitude of the perturbations of the short-wave envelope will be a quantity of the order of $\varepsilon^{1/2}$. Because of the decrease in this amplitude, the contribution from the interaction terms of the next order in ε increases.

When substituting expressions (2.1) and (2.2) into system (1.2), we take into account all terms up to order ε^6 for the zeroth harmonic and $\varepsilon^{9/2}$ for the first harmonic. We shall assume that $L = p_1^{(0)}$, $L_1 = p_2^{(0)}$ and $S = p_1^{(1)}$, $S_1 = p_2^{(1)}$ are functions of $\xi = x_1 - c_g t_1$, $\tau = t_2$ and that L_1 and S_1, by analogy with L and S, evolve during a time τ in accordance with (4.5), that is, the equations

$$\frac{\partial L_1}{\partial \tau} = 0, \quad i\frac{\partial S_1}{\partial \tau} + \beta \frac{\partial^2 S_1}{\partial \xi^2} = 0 \tag{4.6}$$

hold. We also put $p_3^{(0)} = L_2(\xi)$ and $p_3^{(1)} = S_2(\xi)$. Then, using relations (4.4) and (4.5), we obtain the system of equations

$$\frac{\partial L}{\partial \tau} + \varepsilon\left[\chi \frac{\partial^3 L}{\partial \xi^3} + \sigma \frac{\partial L^2}{\partial \xi} - i\lambda\left(S^* \frac{\partial^2 S}{\partial \xi^2} - S\frac{\partial^2 S^*}{\partial \xi^2}\right)\right] = 0 \tag{4.7}$$

$$\frac{\partial S}{\partial \tau} - i\beta \frac{\partial^2 S}{\partial \xi^2} + \varepsilon\left(\eta \frac{\partial^3 S}{\partial \xi^3} + i\gamma S^2 S^* - \mu L \frac{\partial S}{\partial \xi} - \nu S \frac{\partial L}{\partial \xi}\right) = 0$$

All the coefficients of system (4.7) depend solely on the parameter $\varkappa$ and are positive when $\kappa \geqslant 1$

$$c_g = c_e = \frac{\varkappa(3\varkappa+2)}{(3\varkappa+1)(\varkappa+1)^{1/2}}, \quad \sigma = \frac{\varkappa(3\varkappa+2)^3}{4(3\varkappa+1)^3(\varkappa+1)^{1/2}}$$

$$\lambda = \frac{\varkappa^{3/2}}{6(3\varkappa+1)^4(3\varkappa+2)(\varkappa+1)^{3/2}}(27\varkappa^3 + 18\varkappa^2 + 18\varkappa + 4)$$

$$\beta = \frac{3\varkappa^{5/2}(3\varkappa+2)}{2(3\varkappa+1)^3(\varkappa+1)^{3/2}}(3\varkappa^2 + 3\varkappa + 2)$$

$$\eta = \frac{\varkappa^3(3\varkappa+2)}{6(3\varkappa+1)^5(\varkappa+1)^{5/2}}(81\varkappa^4 + 54\varkappa^3 + 81\varkappa^2 + 60\varkappa + 28) \tag{4.8}$$

$$\gamma = \frac{(\varkappa+1)^{1/2}}{12\varkappa^{1/2}(3\varkappa+1)(3\varkappa+2)^2}(9\varkappa - 2), \quad \chi = \frac{\varkappa^3(3\varkappa+2)^5}{6(3\varkappa+1)^5(\varkappa+1)^{5/2}}$$

$$\mu = 2\nu = \frac{\varkappa(3\varkappa+2)}{(3\varkappa+1)^2(\varkappa+1)^{1/2}}$$

Note that the coefficient of the linear term of the first equation of (4.7), χ is identical with the coefficient of the second term of the expansion of the low-frequency branch of dispersion relation (1.3) in the long-wave region (1.4)), while the coefficient of the linear term of the second equation

$$\eta = -\frac{1}{6}\frac{d^3\omega_s}{dk_s^3}$$

If we put $S = 0$, then the system of equations (4.7) reduces to the Korteveg–de Vries equation

$$\frac{\partial L}{\partial \tau_1} + \sigma\frac{\partial L^2}{\partial \xi} + \chi\frac{\partial^3 L}{\partial \xi^3} = 0 \tag{4.9}$$

which describes the propagation of long-wave perturbations in a bubbly liquid. Here, $\tau_1 = t_3 = \varepsilon_t^3$.

The case (l, s) = (1, 1). With this choice, in the case of degeneracy of resonant interaction (4, 4) a non-trivial system of equations (of order ε^2 for the first harmonic and ε^4 for the zeroth approximation also arises). As is usually done, we will assume that

$$p_1^{(0)} = L(\tau,\xi),\quad p_2^{(0)} = L_1(\tau,\xi) \text{ and } p_1^{(1)} = S(\tau,\xi),\quad p_2^{(1)} = S_1(\tau,\xi)$$

We shall also assume that L_1 satisfies the equation

$$\frac{\partial L_1}{\partial \tau} + 2\sigma\frac{\partial LL_1}{\partial \xi} = 0$$

The following interaction model then holds

$$\frac{\partial L}{\partial \tau} + \sigma\frac{\partial L^2}{\partial \xi} + \varepsilon\left[\chi\frac{\partial^3 L}{\partial \xi^3} + \varsigma\frac{\partial L^3}{\partial \xi} + \rho\frac{\partial L|S|^2}{\partial \xi} - i\lambda\left(S^*\frac{\partial^2 S}{\partial \xi^2} - S\frac{\partial^2 S^*}{\partial \xi^2}\right)\right] = 0 \tag{4.10}$$

$$i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} - \gamma S^2 S^* = i\mu L\frac{\partial S}{\partial \xi} + i\nu S\frac{\partial L}{\partial \xi} + \tilde{\delta} L^2 S$$

Here

$$\tilde{\delta} = \frac{(3\varkappa+2)^2}{36\varkappa^{1/2}(3\varkappa+1)(\varkappa+1)^{1/2}}$$

$$\varsigma = \frac{(3\varkappa+2)^3}{36(3\varkappa+1)^5(\varkappa+1)^{3/2}}(27\varkappa^4 + 62\varkappa^3 + 45\varkappa^2 + 7\varkappa - 2)$$

$$\rho = \frac{1}{6(3\varkappa+1)^5(3\varkappa+2)(\varkappa+1)^{1/2}}(27\varkappa^3 + 36\varkappa^2 - 4)$$

The remaining coefficients are determined from (4.8).

If no account is taken of terms of order ε in the first equation of (4.10), the interaction equations will consist of the Hopf equation for L and a non-linear Schrödinger equation with interaction terms for S

$$\frac{\partial L}{\partial \tau} + \sigma\frac{\partial L^2}{\partial \xi} = 0,\quad i\frac{\partial S}{\partial \tau} + \beta\frac{\partial^2 S}{\partial \xi^2} - \gamma S^2 S^* = i\mu L\frac{\partial S}{\partial \xi} + i\nu S\frac{\partial L}{\partial \xi} + \tilde{\delta} L^2 S \tag{4.11}$$

The Hopf equation (the first equation of (4.11)) describes the formation of a shock wave from the initially smooth profile. Hence, if no account is taken of the term of order ε in the first equation of (4.10), the derivative of L with respect to ξ becomes infinite at a certain instant of time. Since, during the formation of a shock wave, terms of order ε, containing the derivatives of L and S with respect to ξ, become large, they begin to have a substantial effect on the dynamics of the long-wave–short-wave interaction.

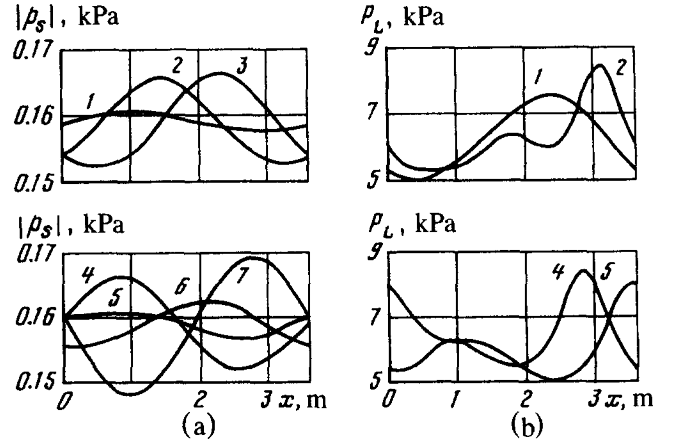

Fig. 3.

5. NUMERICAL INVESTIGATION OF DEGENERATE RESONANCE INTERACTION

The case $(l, s) = (2, 3/2)$. The system of equations (4.7) is integrated numerically using an explicit three-layer scheme with a fourth order of approximation with respect to the coordinate proposed earlier [20] for the numerical solution of the Korteveg–de Vries equation. Periodic boundary conditions and an initial condition of the form

$$L = L_0(1 + \Delta L[1 - \cos\xi]), \quad S = S_0 \tag{5.1}$$

where $L_0 = 5$, $\Delta L = 0.25$, $S_0 = 5$, are used in the integration.

The distributions of the envelope of the short-wave perturbation $|p_S|$ and the profile of the long-wave perturbation p_L ($\varepsilon = 0.1$) with respect to the spatial variable x for water with air bubbles of radius $a_0 = 1$ mm under normal conditions are shown in Fig. 3. Moreover, the parameters of the mixture satisfy the condition of resonant degeneracy ($\Omega_s = 55$ kHz and $\alpha_{g0} = 3.02 \times 10^{-4}$). Curves 1–7 correspond to the following instants of time: $t = 0.019$ s, 0.171 s, 0.209 s, 0.294 s, 0.437 s, 0.532s and 0.722 s. It is clear that a sinusoidal profile is formed in the case of the envelope of the short-wave perturbation. The amplitude of this profile increases up to the instant of time $t = 0.294$ s and then sharply decreases at $t = 0.437$ s, and, after a certain time, it increases again, attaining a maximum value at $t = 0.722$ s (Fig. 3a). In the case of sinusoidal initial conditions, the short wave has no action on the long wave and the long-wave perturbation propagates in accordance with the Korteveg–de Vries equation (4.9). It is important to note that an increase in the amplitude of the short wave occurs during the formation of the second hump in the profile of the long wave (the two-soliton solution of the Korteveg–de Vries equation), and a decrease occurs when it disappear (Fig. 3b). It can be concluded from this that a considerable return flow of energy from the long wave to the short wave is observed during the formation of the two-humped profile, on account of which, in spite of the strong dispersion, the amplitude of the short wave increases. On passing through periods of rise and fall, this amplitude increases on the whole which is seen, for example, when the short-wave distributions for the times $t = 0.209$ s and $t = 0.722$ s (curves 3 and 7 in Fig. 3a) are compared. The essential smallness and transient nature of the non-linear distortions of the short-wave profile should be noted. They manifest themselves most strongly in the reduction in the amplitude of the short wave (see Fig. 4, where curves 1–4 correspond to $t = 0.4557$ s, 0.4652 s, 0.4747 s and 0.4852 s).

The results obtained can be given the following physical interpretation. We assume that high-frequency oscillations as well as low-frequency oscillations are excited in a bubbly liquid. In the general case, these perturbations will propagate in the medium in the form of a wave packet of short waves and a long wave. When the long-wave–short-wave resonance condition is satisfied, interaction occurs which obeys the Zakharov equation (3.5). It is well known that, under the above-mentioned initial-boundary conditions, this system describes the formation of a soliton structure [21]. Hence, at resonance, the form of the short wave is subject to significant non-linear distortions. However, an isolated frequency of the short wave exists in a bubbly liquid, that is, the frequency of the "degeneracy" at which the resonance interaction has a completely different character. First, the long wave propagates independently of the short wave and obeys the Korteveg–de Vries equation. Second, due to the action of the long wave on the short wave, a sinusoidal profile of the short wave envelope is formed which, as it evolves, only experiences insignificant and transient non-linear distortions. This means that practically all the energy of the short wave will be contained in its first harmonic. Third, due to the flow of energy from the long wave to the short wave, there is an increase in the amplitude of the short wave.

The case $(l, s) = (1, 1)$. Equations (4.10) were integrated numerically using the same numerical scheme and boundary conditions as in the case of Eqs (4.7). The initial condition was taken in the form (5.1), where $L_0 = 5$, $\Delta L = 0.05$, $S_0 = 5$.

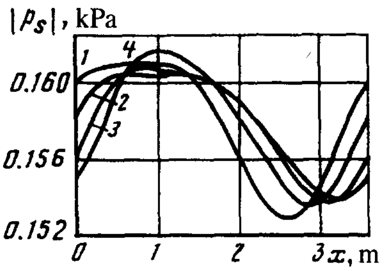

Fig. 4.

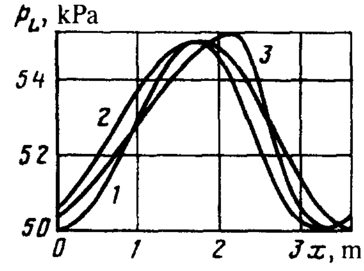

Fig. 5.

The distribution of the long-wave perturbation p_L with respect to x at the instants of time $t = 0$ s, 0.133 s and 0.285 s (curves 1, 2 and 3, respectively) is shown in Fig. 5 (for the same values of the parameters as in the case of model (4.7)). It is clear that the long wave propagates with its profile becoming steeper since, when no account is taken of terms of order ε, the equation for the long wave is the Hopf equation (the first equation of (4.11)).

The subsequent evolution of the long-wave and short-wave perturbations when $\varepsilon = 0.1$ (the thick curve) and $\varepsilon = 0.01$ (the thin curve) is shown in a dimensionless form in Figs 6 and 7. A shock profile is not formed in the case of the long wave. This is due to the contribution of terms of order ε in the first equation of (4.10). Here, if short-wave perturbations are not generated ($S_0 = 0$), the long-wave will propagate in the form of an asymptotically stable wave (the dashed curve in Fig. 6) In the case of a non-zero short-wave amplitude ($S_0 \geqslant 0$), disruption of the long-wave profile occurs. As ε becomes smaller, this disruption becomes more and more significant. A similar process is also observed in the case of the short wave (Fig. 7).

Hence, long-wave–short-wave interaction in the case of resonance degeneracy and equality of the orders of smallnesses of the long-wave and short-wave perturbations (l, s = 1.1) leads to the development of non-linear instability.

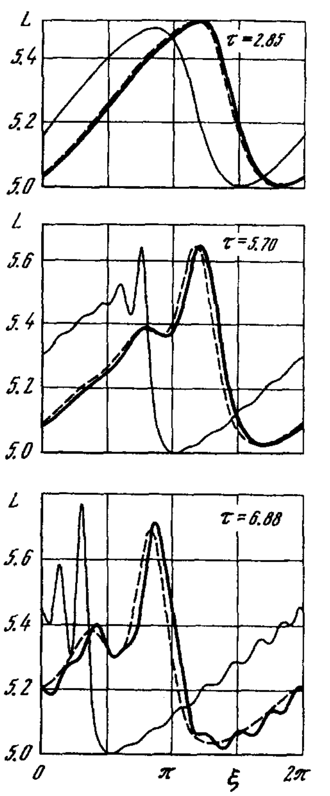

Fig. 6.

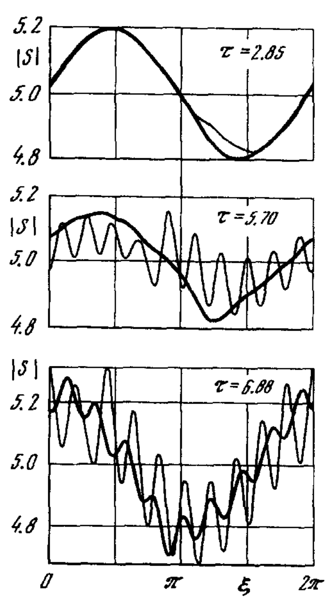

Fig. 7.

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